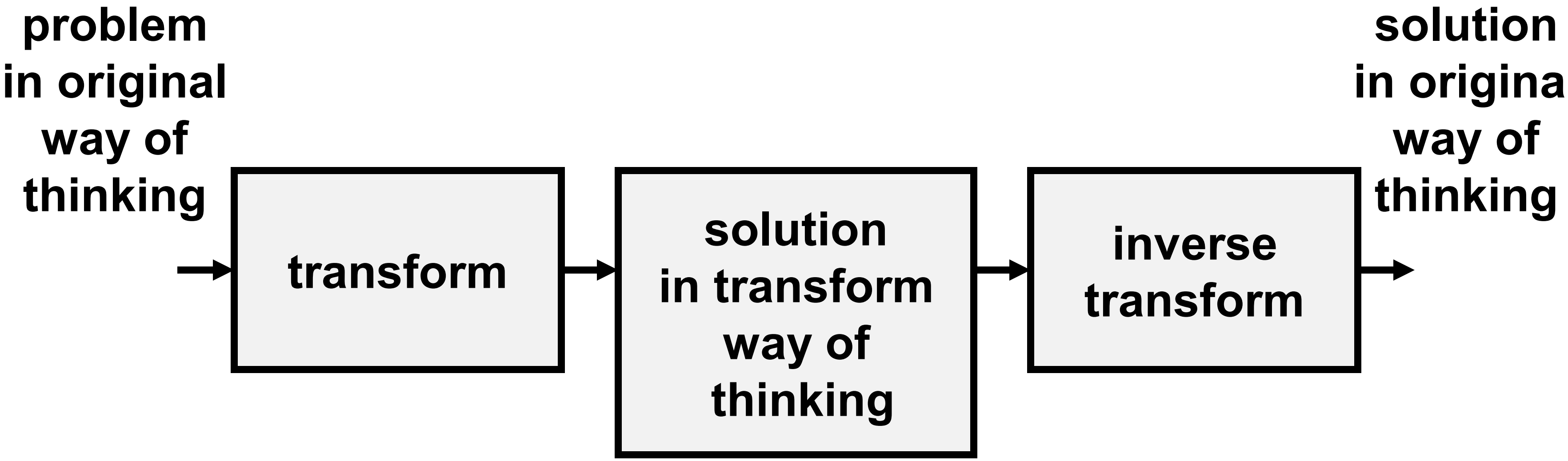
LAPLACE TRANSFORMS

INTRODUCTION

Definition

- Transforms -- a mathematical conversion from one way of thinking to another to make a problem easier to solve



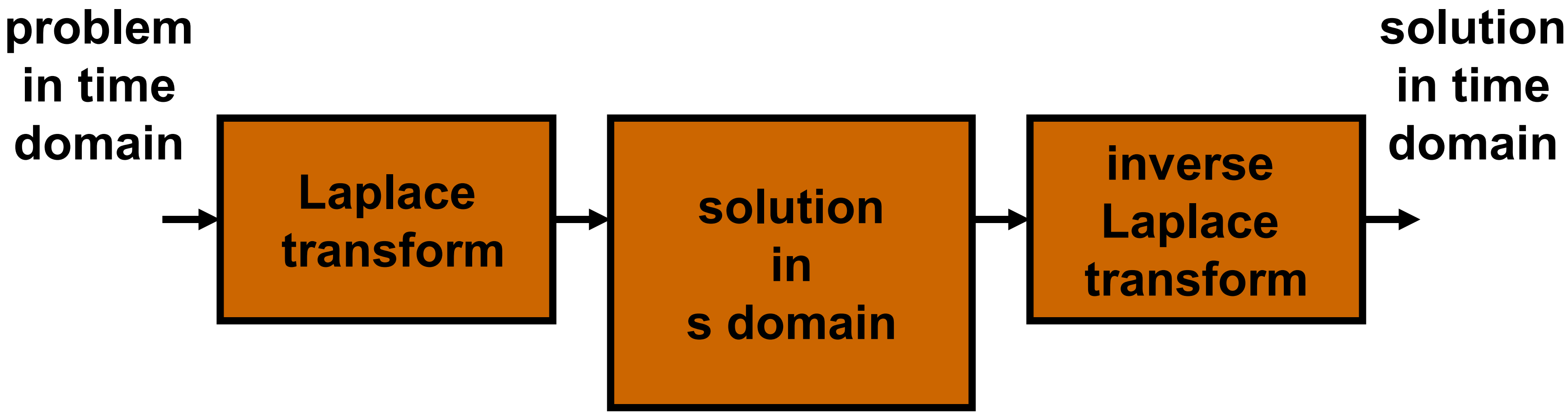
- Transforms
  - Fourier
  - Laplace
  - z-transform
  - wavelets

- **Fourier transforms** are extremely useful in the study of many problems of practical importance involving signals and LTI systems.
  - purely imaginary complex exponentials e^{st}, $s=jw$
  - large class of signals can be represented as a linear combination of complex exponentials
- However, this process applies to any complex number *s*, not just purely imaginary (signals)

Fourier transform >> $F(j\omega)$

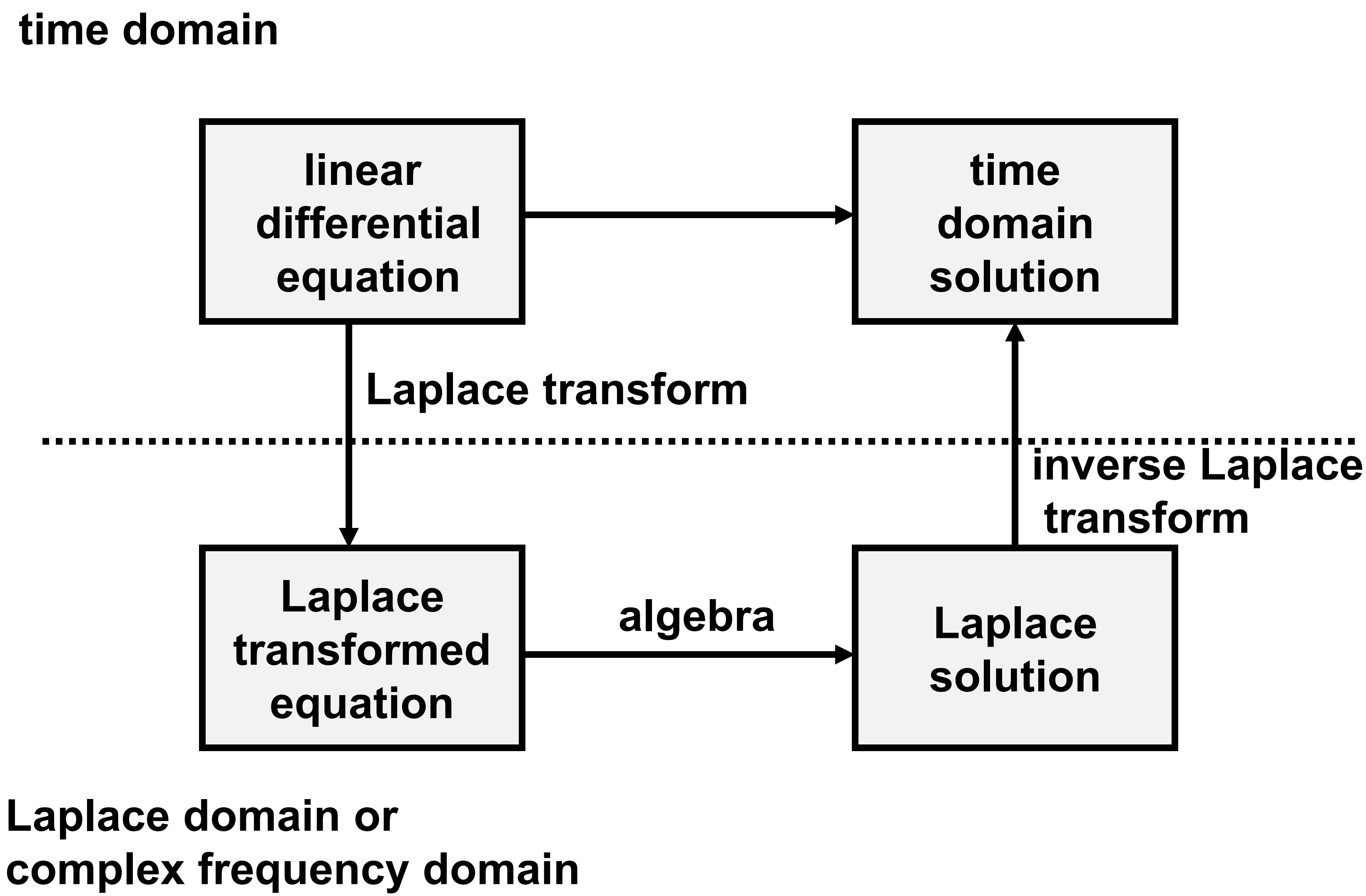
Laplace Transform >> $F(s)$

- This leads to the development of the **Laplace transform** where *s* is an arbitrary complex number.
- Laplace and *z*-transforms can be applied to the analysis of un-stable system (signals with infinite energy) and play a role in the analysis of system stability



- Other transforms
  - Fourier
  - z-transform
  - wavelets

Laplace transformation



Basic Tool For Continuous Time: Laplace Transform

$$\mathbf{L}[f(t)] = F(s) = \int_0^{\infty} f(t)e^{-st}dt$$

- Convert time-domain functions and operations into complex frequency-domain
  - $f(t) \rightarrow F(s) \quad (t \in R,\ s \in C)$
  - Linear differential equations (LDE) $\rightarrow$ algebraic expression in Complex plane
- Discrete systems use the analogous z-transform

The Laplace Transform

The Laplace Transform of a function, f(t), is defined as;

$$L[f(t)] = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

Eq A

The Inverse Laplace Transform is defined by

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{ts} ds$$

Eq B

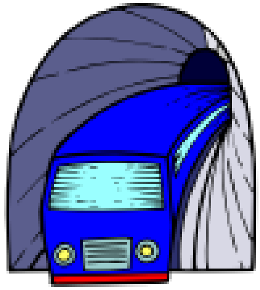
The Laplace Transform

We generally do not use <u>Eq B</u> to take the inverse Laplace. However, this is the formal way that one would take the inverse. To use <u>Eq B</u> requires a background in the use of complex variables and the theory of residues. Fortunately, we can accomplish the same goal (that of taking the inverse Laplace) by using *partial fraction expansion* and recognizing *transform pairs*.

The Laplace Transform

An important point to remember:

$$f(t) \Leftrightarrow F(s)$$



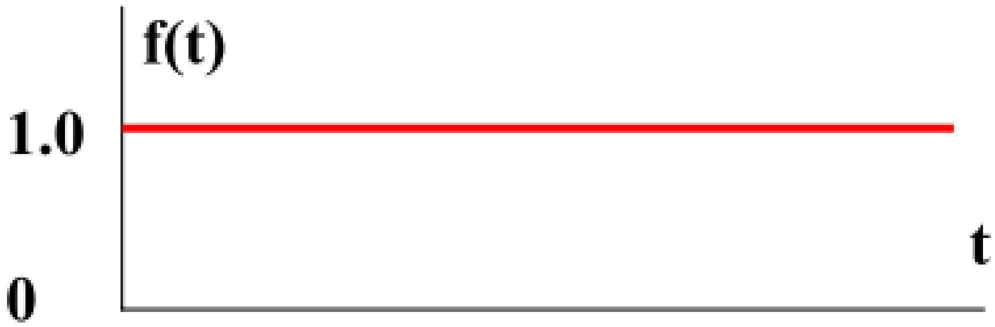
The above is a statement that f(t) and F(s) are transform pairs. What this means is that for each f(t) there is a unique F(s) and for each F(s) there is a unique f(t). If we can remember the Pair relationships between approximately 10 of the Laplace transform pairs we can *go a long way.*

Laplace Transforms of Common Functions

Laplace Transform of the unit step.

$$L[u(t)] = \int_0^\infty 1 e^{-st} dt = \frac{-1}{s} e^{-st} \Big|_0^\infty$$

$$L[u(t)] = \frac{1}{s}$$

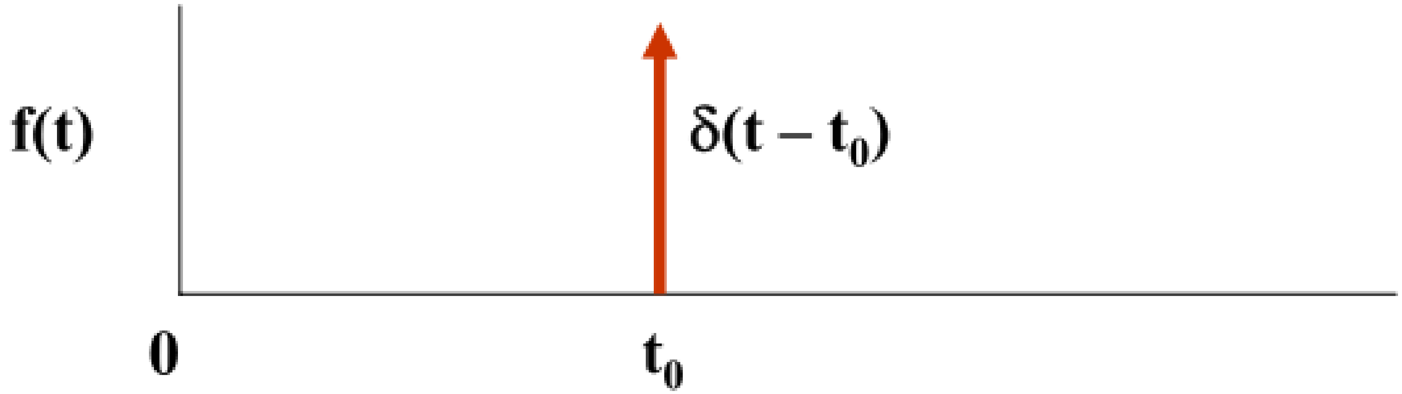


The Laplace Transform of a unit step is: $\boxed{\frac{1}{s}}$

Laplace Transforms of Common Functions

The Laplace transform of a unit impulse:

Pictorially, the unit impulse appears as follows:



Mathematically:

$$\delta(t - t_0) = 0 \quad t \neq 0$$

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t - t_0)\,dt = 1 \qquad \varepsilon > 0$$

Laplace Transforms of Common Functions

The Laplace transform of a unit impulse:

An important property of the unit impulse is a sifting or sampling property. The following is an important.

$$\int_{t_1}^{t_2} f(t)\delta(t-t_0)dt = \begin{cases} f(t_0) & t_1 < t_0 < t_2 \\ 0 & t_0 < t_1, t_0 > t_2 \end{cases}$$

In particular, if we let f(t) = δ(t) and take the Laplace

$$L[\delta(t)] = \int_0^{\infty} \delta(t) e^{-st} dt = e^{-0s} = 1$$

Laplace Transforms of Common Functions

Building transform pairs:

$$L[e^{-at}u(t)] = \int_0^{\infty} e^{-at}e^{-st}\,dt = \int_0^{\infty e} e^{-(s+a)t}\,dt$$

$$L[e^{-at}u(t)] = \frac{-e^{-st}}{(s+a)}\Big|_0^{\infty} = \frac{1}{s+a}$$

A transform pair

$$e^{-at}u(t) \Leftrightarrow \frac{1}{s+a}$$

Laplace Transforms of Common Functions

$$L[tu(t)] = \int_0^{\infty} te^{-st}\,dt$$

$$\int_0^{\infty} u\,dv = uv\Big|_0^{\infty} - \int_0^{\infty} v\,du$$

u = t

dv = e^{-st}dt

$$tu(t) \Leftrightarrow \frac{1}{s^2}$$

A transform pair

Laplace Transforms of Common Functions

Transform Pairs:

f(t)	F(s)
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$(e)^{-at}$	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$

Laplace Transforms of Common Functions

Transform Pairs:

f(t)	F(s)
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin(wt)$	$\frac{w}{s^2+w^2}$
$\cos(wt)$	$\frac{s}{s^2+w^2}$

Laplace Transforms of Common Functions

Transform Pairs:

f(t)	F(s)
$e^{-at}\sin(wt)$	$\dfrac{w}{(s+a)^2+w^2}$
$e^{-at}\cos(wt)$	$\dfrac{s+a}{(s+a)^2+w^2}$
$\sin(wt+\theta)$	$\dfrac{s\sin\theta+w\cos\theta}{s^2+w^2}$
$\cos(wt+\theta)$	$\dfrac{s\cos\theta-w\sin\theta}{s^2+w^2}$

Laplace Transforms Properties

Frequency Shift

$$L[e^{-at} f(t)] = \int_0^{\infty} [e^{-at} f(t)] e^{-st} dt$$

$$= \int_0^{\infty} f(t) e^{-(s+a)t} dt = F(s+a)$$

$$L[e^{-at} f(t)] = F(s+a)$$

Laplace Transforms Properties

Example: Using Frequency Shift

Find the $L[e^{-at}\cos(wt)]$

In this case, f(t) = cos(wt) so,

$$F(s) = \frac{s}{s^2 + w^2}$$

$$and \ \ F(s+a) = \frac{(s+a)}{(s+a)^2 + w^2}$$

$$L[e^{-at}\cos(wt)] = \frac{(s+a)}{(s+a)^2 + (w)^2}$$

Laplace Transforms Properties

Time Integration:

$$L\left[\int_0^\infty f(t)dt\right] = \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$= \frac{1}{s}F(s)$$

Time Differentiation:

If the L[f(t)] = F(s), we want to show:

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

Laplace Transforms Properties

Time Differentiation:

We can extend the previous to show;

$$L\left[\frac{df(t)^2}{dt^2}\right] = s^2F(s) - sf(0) - f'(0)$$

$$L\left[\frac{df(t)^3}{dt^3}\right] = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

general case

$$L\left[\frac{df(t)^n}{dt^n}\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0)$$

$$- \ldots - f^{(n-1)}(0)$$

Linearity	$f_1(t) \pm f_2(t)$	$F_1(s) \pm F_2(s)$
Constant multiplication	$a\ f(t)$	$a\ F(s)$
Complex shift	$e^{at}\ f(t)$	$F(s-a)$
Real shift	$f(t - T)$	$e^{Ts}\ F(as)$
Scaling	$f(t/a)$	$a\ F(as)$

Matlab Code:

```
>> syms f t b
>> f=t*cos(b*t);
>> laplace (f)

ans =

1/(s^2+b^2)^2*(s^2-b^2)
```

Laplace Transforms Theorem

Theorem: **Initial Value Theorem:**

If the function f(t) and its first derivative are Laplace transformable and f(t) Has the Laplace transform F(s), and the $\lim_{s \to \infty} sF(s)$ exists, then

$$\lim_{s \to \infty} sF(s) = \lim_{t \to 0} f(t) = f(0)$$

Initial Value Theorem

The utility of this theorem lies in not having to take the inverse of F(s) in order to find out the initial condition in the time domain. This is particularly useful in circuits and systems.

Laplace Transforms Theorem

Example: **Initial Value Theorem:**

Given;

$$F(s)=\frac{(s+2)}{(s+1)^2+5^2}$$

Find f(0)

$$f(0)=\lim_{s\to\infty} sF(s)=\lim_{s\to\infty} s\frac{(s+2)}{(s+1)^2+5^2}=\lim_{s\to\infty}\left[\frac{s^2+2s}{s^2+2s+1+25}\right]$$

$$=\lim_{s\to\infty}\frac{s^2/s^2+2s/s^2}{s^2/s^2+2s/s^2+(26/s^2)}=1$$

Laplace Transforms Theorem

Theorem: **Final Value Theorem:**

If the function f(t) and its first derivative are Laplace transformable and f(t) has the Laplace transform F(s), and the $\lim_{s \to \infty} sF(s)$ exists, then

$$\lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t) = f(\infty)$$

Final Value Theorem

Again, the utility of this theorem lies in not having to take the inverse of F(s) in order to find out the final value of f(t) in the time domain. This is particularly useful in circuits and systems.

Laplace Transforms Theorem

Example: Final Value Theorem:

Given:

$$F(s)=\frac{(s+2)^2-3^2}{\left[(s+2)^2+3^2\right]} \qquad note\ F^{-1}(s)=te^{-2t}\cos 3t$$

Find $f(\infty)$.

$$f(\infty)=\lim_{s\to 0} sF(s)=\lim_{s\to 0} s\frac{(s+2)^2-3^2}{\left[(s+2)^2+3^2\right]}=0$$

LAPLACE TRANSFORMS

PARTIAL FRACTION EXPANSION

Definition

- Definition -- Partial fractions are several fractions whose sum equals a given fraction
- Purpose -- Working with transforms requires breaking complex fractions into simpler fractions to allow use of tables of transforms

Partial Fraction Expansions

$$\frac{s+1}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

- Expand into a term for each factor in the denominator.

$$\frac{s+1}{(s+2)(s+3)} = \frac{A(s+3) + B(s+2)}{(s+2)(s+3)}$$

- Recombine RHS

$$A+B=1 \qquad 3A+2B=1$$

- Equate terms in s and constant terms. Solve.

$$\frac{s+1}{(s+2)(s+3)} = \frac{-1}{s+2} + \frac{2}{s+3}$$

- Each term is in a form so that inverse Laplace transforms can be applied.

Example of Solution of an ODE (Using Laplace and Inverse Laplace)

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2 \quad y(0) = y'(0) = 0$$

- ODE w/initial conditions

$$s^2 Y(s) + 6s\,Y(s) + 8Y(s) = 2/s$$

- Apply Laplace transform to each term

$$Y(s) = \frac{2}{s(s+2)(s+4)}$$

- Solve for Y(s)

$$Y(s) = \frac{1}{4s} + \frac{-1}{2(s+2)} + \frac{1}{4(s+4)}$$

- Apply partial fraction expansion

$$y(t) = \frac{1}{4} - \frac{e^{-2t}}{2} + \frac{e^{-4t}}{4}$$

- Apply inverse Laplace transform to each term

Different terms of 1st degree

- To separate a fraction into partial fractions when its denominator can be divided into different terms of first degree, assume an unknown numerator for each fraction
- Example --
  - $(11x-1)/(X^2 - 1) = A/(x+1) + B/(x-1)$
  - $= [A(x-1) + B(x+1)]/[(x+1)(x-1)]$
  - $A+B=11$
  - $-A+B=-1$
  - $A=6, B=5$

Repeated terms of 1st degree (1 of 2)

- When the factors of the denominator are of the first degree but some are repeated, assume unknown numerators for each factor
  - If a term is present twice, make the fractions the corresponding term and its second power
  - If a term is present three times, make the fractions the term and its second and third powers

Repeated terms of 1st degree (2 of 2)

- Example --
  - $(x^2+3x+4)/(x+1)^3 = A/(x+1) + B/(x+1)^2 + C/(x+1)^3$
  - $x^2+3x+4 = A(x+1)^2 + B(x+1) + C$
  - $= Ax^2 + (2A+B)x + (A+B+C)$
  - $A=1$
  - $2A+B = 3$
  - $A+B+C = 4$
  - $A=1, B=1, C=2$

Different quadratic terms

- When there is a quadratic term, assume a numerator of the form Ax + B
- Example --
  - $1/[(x+1)(x^2 + x + 2)] = A/(x+1) + (Bx + C)/(x^2 + x + 2)$
  - $1 = A(x^2 + x + 2) + Bx(x+1) + C(x+1)$
  - $1 = (A+B)x^2 + (A+B+C)x + (2A+C)$
  - $A+B=0$
  - $A+B+C=0$
  - $2A+C=1$
  - $A=0.5$, $B=-0.5$, $C=0$

Repeated quadratic terms

- Example --
  - $1/[(x+1)(x^2 + x + 2)^2] = A/(x+1) + (Bx + C)/(x^2 + x + 2) + (Dx + E)/(x^2 + x + 2)^2$
  - $1 = A(x^2 + x + 2)^2 + Bx(x+1)(x^2 + x + 2) + C(x+1)(x^2 + x + 2) + Dx(x+1) + E(x+1)$
  - $A+B=0$
  - $2A+2B+C=0$
  - $5A+3B+2C+D=0$
  - $4A+2B+3C+D+E=0$
  - $4A+2C+E=1$
  - $A=0.25$, $B=-0.25$, $C=0$, $D=-0.5$, $E=0$

LAPLACE TRANSFORMS

SOLUTION PROCESS

Solution process (1 of 8)

- Any nonhomogeneous linear differential equation with constant coefficients can be solved with the following procedure, which reduces the solution to algebra

Solution process (2 of 8)

- Step 1: Put differential equation into standard form
  - $D^2 y + 2D y + 2y = \cos t$
  - $y(0) = 1$
  - $D y(0) = 0$

Solution process (3 of 8)

- Step 2: Take the Laplace transform of both sides
  - $L\{D^2 y\} + L\{2D y\} + L\{2y\} = L\{\cos t\}$

Solution process (4 of 8)

- Step 3: Use table of transforms to express equation in s-domain
  - $L\{D^2 y\} + L\{2D\, y\} + L\{2y\} = L\{\cos \omega\, t\}$
  - $L\{D^2 y\} = s^2 Y(s) - sy(0) - D\, y(0)$
  - $L\{2D\, y\} = 2[\, s\, Y(s) - y(0)]$
  - $L\{2y\} = 2\, Y(s)$
  - $L\{\cos t\} = \mathbf{s/(}s^2 + 1\mathbf{)}$
  - $s^2\, Y(s) - s + 2s\, Y(s) - 2 + 2\, Y(s) = s\, /(s^2 + 1)$

Solution process (5 of 8)

- Step 4: Solve for Y(s)
  - $s^2 Y(s) - s + 2s Y(s) - 2 + 2 Y(s) = \mathbf{s/(}s^2 + 1\mathbf{)}$
  - $(s^2 + 2s + 2) Y(s) = \mathbf{s/(}s^2 + 1\mathbf{)} \mathbf{+} s + 2$
  - $Y(s) = [\mathbf{s/(}s^2 + 1\mathbf{)} \mathbf{+} s + 2]/ (s^2 + 2s + 2)$

    $= (s^3 + 2 s^2 + 2s + 2)/[(s^2 + 1\mathbf{)} (s^2 + 2s + 2)]$

Solution process (6 of 8)

- Step 5: Expand equation into format covered by table
  - $Y(s) = (s^3 + 2s^2 + 2s + 2)/[(s^2 + 1)(s^2 + 2s + 2)]$
  - $= (As + B)/(s^2 + 1) + (Cs + E)/(s^2 + 2s + 2)$
  - $(A+C)s^3 + (2A + B + E)s^2 + (2A + 2B + C)s + (2B + E)$
  - $1 = A + C$
  - $2 = 2A + B + E$
  - $2 = 2A + 2B + C$
  - $2 = 2B + E$
  - $A = 0.2$, $B = 0.4$, $C = 0.8$, $E = 1.2$

Solution process (7 of 8)

- $(0.2s + 0.4)/ (s^2 + 1)$
- $= 0.2\ s/ (s^2 + 1) + 0.4 / (s^2 + 1)$
- $(0.8s + 1.2)/ (s^2 + 2s + 2)$
- $= 0.8\ (s+1)/[(s+1)^2 + 1] + 0.4/ [(s+1)^2 + 1]$

Solution process (8 of 8)

- Step 6: Use table to convert s-domain to time domain
  - $0.2\ s/\ (s^2 + 1)$ becomes $0.2 \cos t$
  - $0.4\ /\ (s^2 + 1)$ becomes $0.4 \sin t$
  - $0.8\ (s+1)/[(s+1)^2 + 1]$ becomes $0.8\ e^{-t} \cos t$
  - $0.4/\ [(s+1)^2 + 1]$ becomes $0.4\ e^{-t} \sin t$
  - $y(t) = 0.2 \cos t + 0.4 \sin t + 0.8\ e^{-t} \cos t + 0.4\ e^{-t} \sin t$

Partial Fraction Expansion Using Matlab

$$\frac{8x^2+3x-21}{x^3-7x-6} = \frac{8x^2+3x-21}{(x+2)(x-3)(x+1)}$$

```
b=[8 3 -21];
a=[1 0 -7 -6];
[r,p,k]=residue(b,a)
```

$$\frac{b(s)}{a(s)} = \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \ldots + \frac{r_n}{s-p_n} + k(s)$$

```
r =

    3.0000
    1.0000
    4.0000


p =

    3.0000
   -2.0000
   -1.0000


k =

    []
```

$$\frac{8x^2+3x-21}{(x+2)(x-3)(x+1)} = \frac{1}{x+2} + \frac{3}{x-3} + \frac{4}{x+1}$$

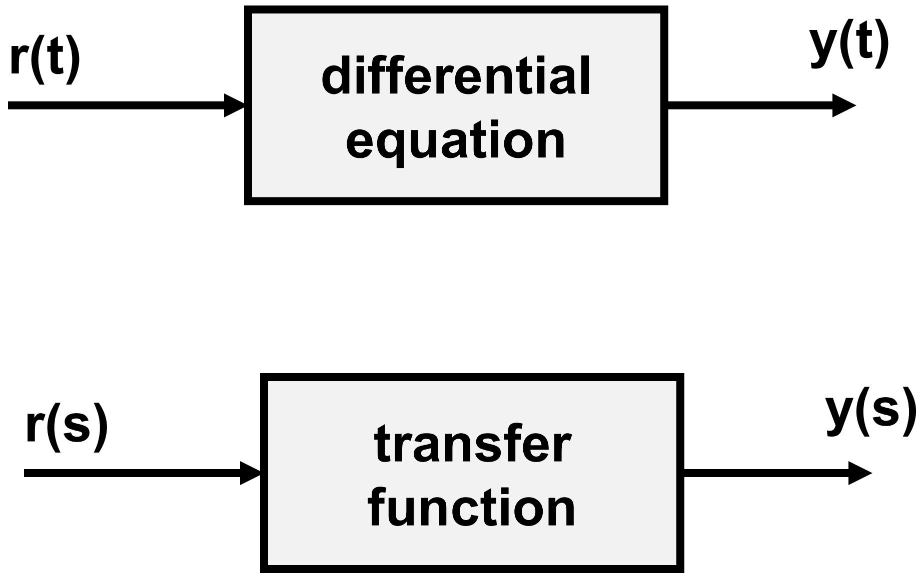
We can also use <u>ilaplace (F);</u> but the result may not be simplified!

LAPLACE TRANSFORMS

TRANSFER FUNCTIONS

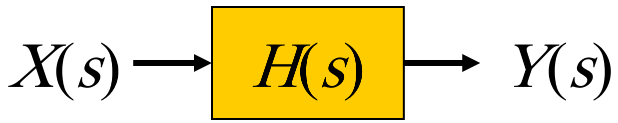
Introduction

- Definition -- a transfer function is an expression that relates the output to the input in the s-domain



Transfer Function

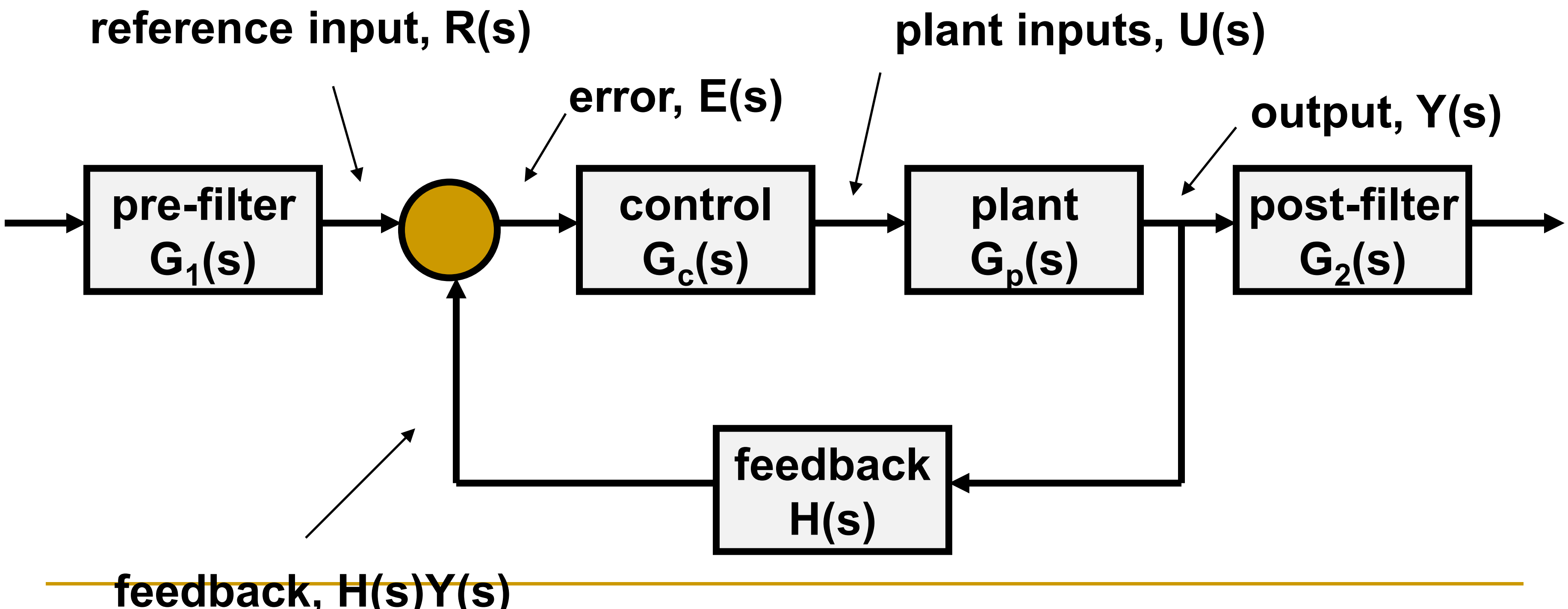
- Definition
  - H(s) = Y(s) / X(s)



- Relates the output of a linear system (or component) to its input
- Describes how a linear system responds to an impulse
- All linear operations allowed
  - Scaling, addition, multiplication

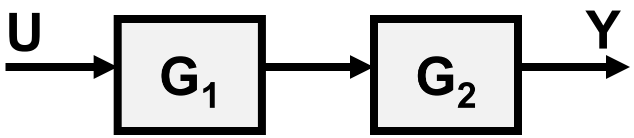
block diagram

- Pictorially expresses flows and relationships between elements in system
- Can simplify according to rules

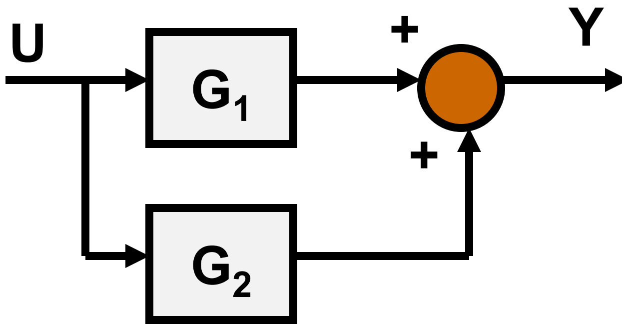


Block diagram reduction rules

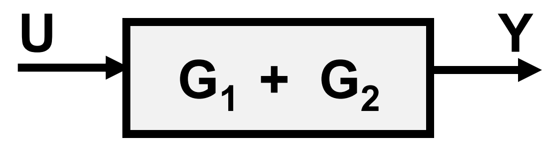
Series



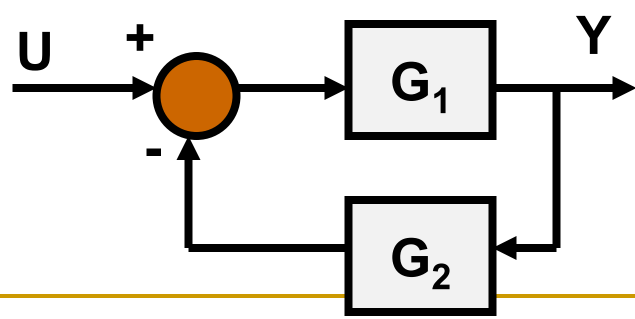
Parallel

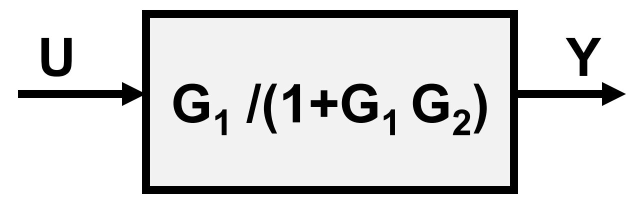


$H(s)$

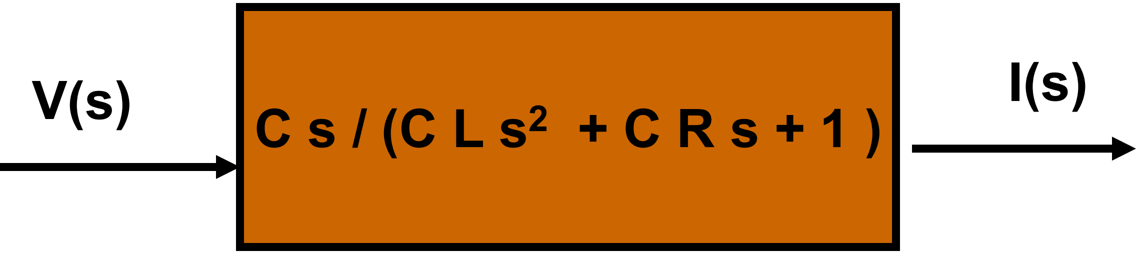


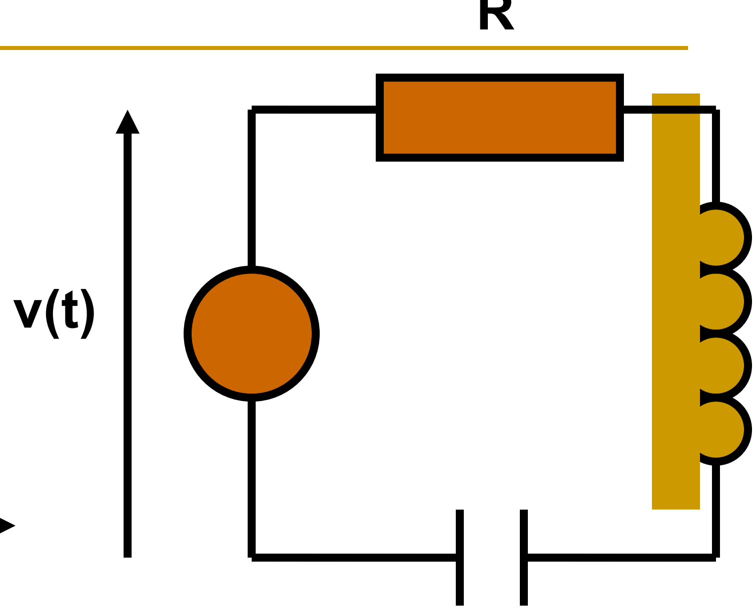
Feedback





Example





- V(s)
  - = (R + 1/(C s) + s L) I(s)
  - = (C L s^2 + C R s + 1)/(C s) I(s)
- I(s)/V(s) = C s / (C L s^2 + C R s + 1)

C

$$v(t) = R\, I(t) + 1/C \int I(t)\, dt + L\, di(t)/dt$$

$$V(s) = [R\, I(s) + 1/(C\, s)\, I(s) + s\, L\, I(s)]$$

Note: Ignore initial conditions